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Double Harmonic Mappings of Riemannian Manifolds and Its Applications to Stationary Axisymmetric Gravitational Fields

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Harmonic mappings of Riemannian manifolds are discussed by a double complex function method, and the double-complex Ernst equation and the related Bäcklund transformations are naturally derived. Further, the Ernst solution and its dual solution are obtained by two different methods, respectively. Therefore, the results obtained by A. Eris are extended to a double form.

1. INTRODUCTION AND PRELIMINARIES

Eris (1977; Eris and Nutku, 1975) investigated applications of harmonic mappings of Riemannian manifolds to general relativity. In particular, for stationary axisymmetric gravitational fields, the Ernst solution is concretely derived by using the Hamilton–Jacobi (HJ) technique and composite mappings. However, only ordinary complex functions are used in the method, and therefore half of the complete results are lost. In fact, Zhong (1985) has given a double complex function method combining ordinary complex numbers with hyperbolic complex numbers, and established the double-complex Ernst equation. By using this method the solutions for the gravitational fields are always obtained in pairs. Thus it should be possible to apply the double-complex function method to harmonic mappings of Riemannian manifolds. The purpose of this paper is to discuss harmonic mappings by the double-complex function method, and extend Eris' results to a double form. We find that if we take some kind of double manifolds to discuss its harmonic mappings, we can naturally derive the double-complex Ernst equation and related Bäcklund transformations

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(BT). Further, the double Ernst solution, i.e., the Ernst solution (Ernst, 1968) and its dual solution, are respectively obtained by two different methods, one using the class of double BTs given in Section 3, the other is extending Eris' results to a double form.

For the sake of convenience, some necessary results and notation (Zhong, 1985; Eris and Nutku, 1975; Eris, 1977) are collected here. Let J denote the double imaginary unit, i.e., $J = i$ $(i^2 = -1)$ or $J = \epsilon$ $(\epsilon^2 = 1, \epsilon \neq 1)$. Let all a_n be real numbers, and the series $\sum_{n=0}^{\infty} |a_m|$ be convergent; then

$$a(J) = \sum_{n=0}^{\infty} a_n J^{2n} \tag{1}$$

is called a double-real number. If $a(J)$ and $b(J)$ are both double-real numbers, $Z(J) = a(J) + J \cdot b(J)$ is called a double-complex number. Sometimes $Z(J)$ may be directly written as $Z(J) = (Z_C, Z_H)$, where

$$Z_C = Z(J = i), \qquad Z_H = Z(J = \epsilon) \tag{2}$$

Similarly, if $U(x, y; J)$ and $V(x, y; J)$ are both double-real functions of variables x, y, then $F(x, y; J) = U(x, y; J) + J \cdot V(x, y; J)$ is called a double-complex function. Further, if $F(x, y; J)$ is a double-complex analytic function, double Cauchy–Riemann (CR) conditions satisfied by $F(x, y; J)$ can be derived as follows:

$$\frac{\partial U(J)}{\partial x} = \frac{\partial V(J)}{\partial y}, \qquad \frac{\partial U(J)}{\partial y} = J^2 \cdot \frac{\partial V(J)}{\partial x} \tag{3}$$

When $J = i$, equation (3) is just changed into the ordinary CR conditions

$$\frac{\partial U_C}{\partial x} = \frac{\partial V_C}{\partial y}, \qquad \frac{\partial U_C}{\partial y} = -\frac{\partial V_C}{\partial x} \tag{4a}$$

When $J = \epsilon$, equation (3) is changed into new equations called hyperbolic CR conditions

$$\frac{\partial U_H}{\partial x} = \frac{\partial V_H}{\partial y}, \qquad \frac{\partial U_H}{\partial y} = \frac{\partial V_H}{\partial x} \tag{4b}$$

Therefore, for a double-complex analytic function $F(J)$, one can write

$$F(J) = F(Z(J)) = F(x + J \cdot y) \tag{5}$$

Let M and M' be two ordinary Riemannian manifolds with the metric

$$\begin{aligned} dl^2 &= g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \qquad (\mu, \nu = 1, 2, \ldots, n) \\ dL^2 &= G_{AB}(\phi)\, d\phi^A\, d\phi^B \qquad (A, B = 1, 2, \ldots, m) \end{aligned} \tag{6}$$

respectively, and a mapping $\phi: M \to M'$. If the mapping ϕ makes the action

$$I = \int d^n x \sqrt{g}\, g^{\mu\nu}(x) \frac{\partial \phi^A}{\partial x^\mu} \frac{\partial \phi^B}{\partial x^\nu} G_{AB}(\phi) \tag{7}$$

satisfy the condition $\delta I = 0$, it is called harmonic. The necessary and sufficient conditions for a mapping to be harmonic are given by the Euler equations

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left[\sqrt{g}\, g^{\mu\nu} \frac{\partial \phi^C}{\partial x^\nu} \right] + g^{\mu\nu} \Gamma^C_{AB} \frac{\partial \phi^A}{\partial x^\mu} \frac{\partial \phi^B}{\partial x^\nu} = 0 \tag{8}$$

where $C = 1, 2, \ldots, m$. The Γ^C_{AB} are coefficients of the Riemannian connection on M'. Notice that the above Euler equations are nonlinear partial differential equations of variables ϕ^A. The condition for two different coordinates ϕ' and ϕ on M' to keep the Euler equations of the same form is

$$G'_{AB}(\phi') = G_{AB}(\phi') \tag{9}$$

This means $G'_{AB}(\phi')$ and $G_{AB}(\phi)$ must be the same in functional form for the variables ϕ' and ϕ. Condition (9) may be expressed as

$$G'_{AB}(\phi') = \frac{\partial \phi^C}{\partial \phi'^A} \frac{\partial \phi^D}{\partial \phi'^B} G_{CD}(\phi), \qquad \phi = \phi(\phi') \tag{10}$$

In fact, equation (10) implies functional relations between two different solutions for the Euler equation (8), i.e., Bäcklund transformations (BT).

2. A CLASS OF DOUBLE BÄCKLUND TRANSFORMATIONS FOR THE EULER EQUATIONS

Now we consider a double-two-dimensional manifold $M'(J)$ with the metric

$$dL^2(J) = G_{AB}(J)\, d\phi^A(J)\, d\phi^B(J) \tag{11}$$

and take the metric as diagonal, i.e., $G_{11} = \alpha(\phi)$, $G_{22} = -J\beta(\phi)$, and $G_{12} = G_{21} = 0$, where $\alpha(\phi)$ and $\beta(\phi)$ are real functions of ϕ, and ϕ is a double-real function of coordinate x on M, i.e., $\phi = \phi(x; J)$. If we let $\phi = (\phi^1, \phi^2)$ and $\Phi = (\Phi^1, \Phi^2)$ denote two different double solutions of equation (8), then equation (10) results in

$$\alpha(\Phi)\left(\frac{\partial \Phi^1}{\partial \phi^1}\right)^2 - J^2 \beta(\Phi)\left(\frac{\partial \Phi^2}{\partial \phi^1}\right)^2 = \alpha(\phi) \tag{12}$$

$$\alpha(\Phi)\left(\frac{\partial \Phi^1}{\partial \phi^2}\right)^2 - J^2 \beta(\Phi)\left(\frac{\partial \Phi^2}{\partial \phi^2}\right) = -J^2 \beta(\phi) \tag{13}$$

$$\alpha(\Phi)\left(\frac{\partial \Phi^1}{\partial \phi^1}\right)\left(\frac{\varphi \Phi^1}{\partial \phi^2}\right) - J^2 \beta(\Phi)\left(\frac{\partial \Phi^2}{\partial \phi^1}\right)\left(\frac{\partial \Phi^2}{\partial \phi^2}\right) = 0 \tag{14}$$

Furthermore, if $\alpha(\phi) = \beta(\phi)$, the metric (11) changes into

$$dL^2(J) = \alpha(\phi)[(d\phi^1)^2 - J^2(d\phi^2)^2] \tag{15}$$

When $J = i$, we have

$$dL_C^2 = \alpha(\phi_C)[(d\phi_C^1)^2 + (d\phi_C^2)^2] \tag{16a}$$

When $J = \epsilon$, equation (15) is changed into

$$dL_H^2 = \alpha(\phi_H)[(d\phi_H^1)^2 - (d\phi_H^2)^2] \tag{16b}$$

Thus, equations (12)–(14) can be rewritten as

$$\frac{\partial \Phi^1}{\partial \phi^1} = \frac{\partial \Phi^2}{\partial \phi^2}; \qquad \frac{\partial \Phi^1}{\partial \phi^2} = J^2\left(\frac{\partial \Phi^2}{\partial \phi^1}\right) \tag{17}$$

which are just the double CR conditions. Further, if Φ^1 and Φ^2 have continuous derivatives to ϕ^1 and ϕ^2, respectively, $\Phi^1 + J \cdot \Phi^2$ is also an analytic function of the double-complex variable $\phi^1 + J \cdot \phi^2$. In this case, a class of double BTs of the Euler equation (8) satisfies the following double analytic function relation:

$$W(J) = F[w(J)] \tag{18}$$

where $W(J) = \Phi^1 + J \cdot \Phi^2$ and $w(J) = \phi^1 + J \cdot \phi^2$.

Now we explain the meaning of the above results. It is well known that the metric of an arbitrary two-dimensional manifold is either positive definite [the signature is $(+1, +1)$; we consider no distinction between a negative-definite and a positive-definite metric] or indefinite [the signature is $(+1, -1)$]. Evidently, by virtue of equation (15) we give the most common results. The metric corresponding to $J = i$ is positive definite, and in this case the class of BTs for Euler equations satisfies the ordinary CR conditions (4a). This indicates that there is an ordinary-complex analytic relation between any two solutions of the Euler equations. But the metric corresponding to $J = \epsilon$ is indefinite, and under the condition, the class of BTs for the Euler equations satisfies the hyperbolic CR conditions (4b). Hence, there is a hyperbolic-complex analytic relation between any two solutions for the Euler equations. Evidently, this is a new result. It follows that a class of double BTs for the Euler equations can be obtained by doubling harmonic mappings, and they satisfy double CR conditions.

3. DOUBLE-COMPLEX ERNST EQUATION AND RELATED BÄCKLUND TRANSFORMATIONS

In order to apply the above results to the general theory of relativity, we let M denote a three-dimensional Riemannian manifold with the metric

$$dl^2 = d\rho^2 + dz^2 + \rho^2\, d\varphi^2 \tag{19}$$

and let M' denote a double-two-dimensional Riemannian manifold with the metric

$$dL^2 = F^{-2}(J)[dF^2(J) - J^2\, d\Omega^2(J)] \tag{20}$$

where $F(J)$ and $\Omega(J)$ are double-real functions of ρ and z, respectively. It is easily verified that if we take the action

$$I = \int F^2(J)[(\nabla F)^2 - J(\nabla\Omega)^2]\rho\, d\rho\, dz\, d\varphi \tag{21}$$

then the Euler equation corresponding to $\delta I = 0$ is

$$\mathrm{Re}(E(J))\, \nabla^2 E(J) = \nabla E(J) \cdot \nabla E(J) \tag{22}$$

where $E(J) = F(J) + J \cdot \Omega(J)$ is a double-complex Ernst potential, $\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2$, and $\nabla = (\partial_\rho, \partial_z)$. It is clear that equation (22) is just the double-complex Ernst equation (Zhong, 1985). When $J = i$, we have

$$\mathrm{Re}\, E_C\, \nabla^2 E_C = \nabla E_C \cdot \nabla E_C \tag{23}$$

which is just the result given by Eris (1977). When $J = \epsilon$, equation (22) is changed into

$$\mathrm{Re}\, E_H\, \nabla^2 E_H = \nabla E_H \cdot \nabla E_H$$

This is the hyperbolic-complex Ernst equation.

Let the metric for stationary axisymmetric gravitational fields be

$$ds^2 = f(dt - \omega\, d\varphi)^2 - f^{-1}[e^\gamma(dz^2 + d\rho^2 + \rho^2\, d\varphi^2)] \tag{24}$$

where (ρ, z, φ) denote the cylindrical coordinates, and f, ω, and γ are real functions of ρ and z only. If $E(J) = F(J) + J \cdot \Omega(\mathrm{J})$ is a solution of equation (22), two different physical (real) solutions (f, ω) and $(\hat{f}, \hat{\omega})$ can be obtained as follows:

$$\begin{aligned} (f, \omega) &= (F_C, V_{F_C}^{-1}(\Omega_C)) \\ (\hat{f}, \hat{\omega}) &= (T(F_H), \Omega_H)) \end{aligned} \tag{25}$$

where the NK transformations (Neugebauer and Kramer, 1969) are

$$\begin{aligned} &T: f \to T(f) = \frac{\rho}{f} \\ &V_f: \omega \to \Psi = V_f(\omega) \\ &\Psi = \int \frac{f^2}{\rho}(\partial_z\omega\, d\rho - \partial_\rho\omega\, dz) \end{aligned} \tag{26}$$

It follows that finding solutions of stationary axisymmetric gravitational fields can be focused on finding solutions of the double-complex Ernst equation (22). From equations (21) and (22), it is seen that the double-complex Ernst equation is the natural result to double harmonic mappings. Hence, the class of BTs for the Euler equations should be fit for the double-complex Ernst equation. If we let $E'(J) = F(J) + J \cdot \Omega(J)$ and $E(J) = U(J) + J \cdot V(J)$ be two double solutions of equation (22), then according to equations (12)–(14) and (20), a class of double BTs $T(J)$: $E(J) \to E'(J) = T(J)E(J)$ can be written as

$$U^2(J)\left[\left(\frac{\partial F(J)}{\partial U(J)}\right)^2 - J^2\left(\frac{\partial \Omega(J)}{\partial U(J)}\right)^2\right] = F^2(J) \tag{27}$$

$$U^2(J)\left[\left(\frac{\partial F(J)}{\partial V(J)}\right)^2 - J^2\left(\frac{\partial \Omega(J)}{\partial V(J)}\right)^2\right] = -J^2F(J) \tag{28}$$

$$\frac{\partial F(J)}{\partial U(J)} \cdot \frac{\partial F(J)}{\partial V(J)} - J^2\frac{\partial \Omega(J)}{\partial U(J)} \cdot \frac{\partial \Omega(J)}{\partial V(J)} = 0 \tag{29}$$

Furthermore equation (27)–(29) can be simplified as

$$\frac{\partial F(J)}{\partial U(J)} = \frac{\partial \Omega(J)}{\partial V(J)}, \qquad \frac{\partial F(J)}{\partial V(J)} = J^2\frac{\partial \Omega(J)}{\partial U(J)} \tag{30}$$

Evidently, the double BT $T(J)$ for the double-complex Ernst equation also satisfies the double CR conditions. Therefore, there is a double-complex analytic relation between $E'(J)$ and $E(J)$. From equations (27) and (30), we obtain the concrete form of the double BT $T(J)$ as follows:

$$\frac{dE'(J)}{dE(J)} = \frac{F(J)}{U(J)}e^{J\theta} \tag{31}$$

where the double exponential function is

$$e^{J\theta} = \sum_{n=0}^{\infty} \frac{1}{n!}(\theta J)^n = C(\theta J) + J \cdot S(\theta J)$$

and $C(\theta J)$ and $S(\theta J)$ denote the double cosine and sine functions, respectively (Zhong, 1985). When $J = i$, equation (32) is an ordinary BT T_C for the ordinary-complex Ernst equation (23a),

$$T_C: E_C \to E'_C, \qquad \frac{dE'_C}{dE_C} = \frac{F_C}{U_C}e^{i\theta} \tag{32}$$

When $J = \epsilon$, equation (32) is a hyperbolic BT T_H for the hyperbolic-complex Ernst equation (23b),

$$T_H: E_H \to E'_H, \qquad \frac{dE'_H}{dE_H} = \frac{F_H}{U_H}e^{\epsilon\theta} \tag{33}$$

where the hyperbolic exponential function $e^{\epsilon\theta} = \text{ch}\,\theta + J \cdot \text{sh}\,\theta$. In fact, we can give examples: (i)

$$E'(J) = \frac{1}{E(J)} \tag{34}$$

(ii) (Zhang, 1985)

$$\begin{aligned} E'(J) &= \frac{a(J)E(J) + J \cdot b(J)}{J \cdot c(J)E(J) + d(J)} \\ a(J)d(J) &- J^2 b(J)c(J) = 1 \end{aligned} \tag{35}$$

(iii) (Wu, 1992)

$$\begin{gathered} \partial_p E'(J) = \frac{F'(J)}{F(J)} g(J)\partial_p E(J) \\ \partial_q E'(J) = \frac{F'(J)}{F(J)} g(J)\partial_q E(J) \\ g(J)\bar{g}(J) = 1 \\ p = \rho + J \cdot z, \qquad q = \rho - J \cdot z \end{gathered} \tag{36}$$

Among them every pair of $(E(J), E'(J))$ can be verified to meet equation (32). However, it is noticed that, by using a dual mapping, the class of double BTs $T(J)$ may be changed into a new class of double BTs $T'(J)$ which no longer satisfies equation (32) (Wu, 1992). This means the double BT $T(J)$ discussed above is only one class of many possible double BTs for the double-complex Ernst equation.

4. A DOUBLE STATIONARY SOLUTION OF THE DOUBLE-COMPLEX ERNST EQUATION

Let $E'(J) = U$ denote a static potential [to the static potential $E(J)$, $E(J) = E_C = E_H = U$], which is an ordinary real function of ρ and z. Thus, U satisfies

$$U\,\nabla^2 U = \nabla U \cdot \nabla U \tag{37}$$

If we take $U = e^{2\sigma}$, then equation (37) can be simplified as the Laplace equation

$$\nabla^2 \sigma = 0 \tag{38}$$

In addition, let $E(J) = F(J) + J \cdot \Omega(J)$ denote a stationary solution of the double-complex Ernst equation (22); thus the relations between $E(J)$ and

$E'(J)$ may be expressed as

$$F(J) = F(U, J), \qquad \Omega(J) = \Omega(U, J) \tag{39}$$

By using equation (39), one changes the double-complex Ernst equation into

$$\begin{aligned} \frac{d^2F(J)}{d\sigma^2} - \frac{1}{F(J)}\left[\left(\frac{dF(J)}{d\sigma}\right)^2 + J^2\left(\frac{d\Omega(J)}{d\sigma}\right)^2\right] = 0 \\ \frac{d^2\Omega(J)}{d\sigma^2} - \frac{2}{F(J)}\frac{dF(J)}{d\sigma}\cdot\frac{d\Omega(J)}{d\sigma} = 0 \end{aligned} \tag{40}$$

When $J = i$, we obtain

$$\begin{aligned} \frac{d^2F_C}{d\sigma^2} - \frac{1}{F_C}\left[\left(\frac{dF_C}{d\sigma}\right)^2 - \left(\frac{d\Omega_C}{d\sigma}\right)^2\right] = 0 \\ \frac{d^2\Omega_C}{d\sigma^2} - \frac{2}{F_C}\frac{dF_C}{d\sigma}\cdot\frac{d\Omega_C}{d\sigma} = 0 \end{aligned} \tag{41a}$$

These are just the geodesic equations with the affine parameter σ on M' (Eris, 1977). When $J = \epsilon$, equations (40) change into

$$\begin{aligned} \frac{d^2F_H}{d\sigma^2} - \frac{1}{F_H}\left[\left(\frac{dF_H}{d\sigma}\right)^2 + \left(\frac{d\Omega_H}{d\sigma}\right)^2\right] = 0 \\ \frac{d^2\Omega_H}{d\sigma^2} - \frac{2}{F_H}\frac{dF_H}{d\sigma}\cdot\frac{d\Omega_H}{d\sigma} = 0 \end{aligned} \tag{41b}$$

These are the geodesic equations on M'_H. From equations (40) we obtain the solution which satisfies the conditions $\Omega(J) = 0$ and $F(J) = 1$ at infinity,

$$[\Omega(J) + b(J)]^2 - J^2F^2(J) = a^2(J) \tag{42}$$

where $a(J)$ and $b(J)$ are both double-real constants and satisfy

$$b^2(J) - J^2 = a^2(J) \tag{43}$$

From equation (42), it is seen that when $J = i$, we have

$$(\Omega_C + b_C)^2 + F_C^2 = a_C^2 \tag{44a}$$

This indicates that the solution (F_C, Ω_C) of the ordinary-complex Ernst equation (23a) corresponds to a cycle on M'_C, but when $J = \epsilon$, equation (42) changes into

$$(\Omega_H + b_H)^2 - F_H^2 = a_H^2 \tag{44b}$$

which implies that the solution (F_H, Ω_H) of the hyperbolic-complex Ernst equation (23b) corresponds to a hyperbola on M'_H. In addition, if we rewrite equation (27) as

$$\left(\frac{dF(J)}{dU}\right)^2 - J^2\left(\frac{d\Omega(J)}{dU}\right)^2 = \frac{F^2(J)}{U^2} \tag{45}$$

and combine equation (45) with equations (42) and (43), then the double BT between the stationary solution $(F(J), \Omega(J))$ and the static solution U can be derived as follows:

$$\begin{aligned} F(J) &= \frac{2a(J)U}{a(J)(U^2+1) - b(J)(U^2-1)} \\ \Omega(J) &= -\frac{U^2-1}{a(J)(U^2+1) - b(J)(U^2-1)} \end{aligned} \tag{46}$$

If we let

$$\begin{aligned} a(J) &= \mathrm{CS}[\theta J] = \begin{cases} \csc\theta, & (J=i) \\ \mathrm{csch}\,\theta, & (J=\epsilon) \end{cases} \\ b(J) &= \mathrm{CT}[\theta J] = \begin{cases} \mathrm{ctg}\,\theta, & (J=i) \\ \mathrm{cth}\,\theta, & (J=\epsilon) \end{cases} \end{aligned} \tag{47}$$

where the double cosecant function is $\mathrm{CS}(\theta J) = (\mathrm{S}(\theta J))^{-1}$ and the double cotangent function is $\mathrm{CT}(\theta J) = \mathrm{C}(\theta J)/\mathrm{S}(\theta J)$, then, obviously, $a(J)$ and $b(J)$ meet equation (43). Thus, equation (46) can be written as

$$\begin{aligned} F(J) &= \frac{2U}{(U^2+1) - (U^2-1)\,\mathrm{C}[\theta J]} \\ \Omega(J) &= \frac{(U^2-1)\,\mathrm{S}[\theta J]}{(U^2+1) - (U^2-1)\,\mathrm{C}[\theta J]} \end{aligned} \tag{48}$$

which is just the solution of the double-complex Ernst equation (22), i.e.,

$$\xi(J) = \frac{1+E(J)}{1-E(J)} = -e^{J\theta}\,\mathrm{cth}\, s, \qquad U = e^{2\sigma} \tag{49}$$

where $E(J) = F(J) + J \cdot \Omega(J)$, and $\nabla^2\sigma = 0$. When $J = i$, we obtain

$$E(J) = E_C = F_C + i\Omega_C, \qquad \xi_C = -e^{i\theta}\,\mathrm{cth}\,\sigma \tag{50a}$$

This is just the well-known Ernst solution (Ernst, 1968). But when $J = \epsilon$, we have

$$E(J) = E_H = F_H + \epsilon\Omega_H, \qquad \xi_H = -e^{\epsilon\theta}\,\mathrm{cth}\,\sigma \tag{50b}$$

This is the dual solution of (50), which is new.

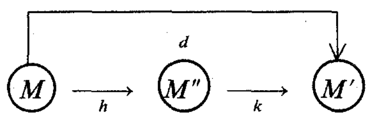
So far in this work we have naturally derived the double-complex Ernst solution $\xi(J) = -e^{J\theta} \operatorname{cth} \sigma$, which is a double BT of the double-complex Ernst equation between a double stationary solution and a static solution. Evidently, this result once again demonstrates that there is, indeed, a class of double BTs which meets the double CR conditions for some solutions of the double-complex Ernst equation.

5. A DOUBLE FORM OF ERIS' RESULTS

5.1. Double Composite Mappings

Eris (1977) applied the HJ technique and composite mappings to stationary axisymmetric gravitational fields in order to generate new class of solutions. But, in fact, only the Ernst solution was obtained by this method. In order to obtain a new solution which is a dual solution of the Ernst solution, now we will extend Eris' results to a double form.

Let d denote a mapping from Riemannian manifold M to M'. If we introduce a third Riemannian manifold M'' which is one-dimensional as follows,



(51)

then the mapping $d: M \to M'$ can be regarded as a composite mapping

$$d = k \circ h \tag{52}$$

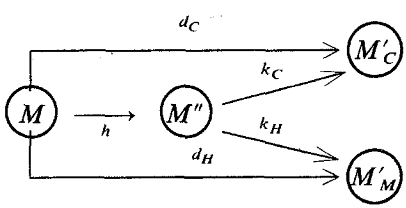
where $h: M \to M''$ and $k: M'' \to M'$. If the manifolds M, M'', and $M'(J)$ have the metrics

$$dl^2 = d\rho^2 + dz^2 + \rho^2\, d\varphi^2 \tag{53a}$$

$$dl''^2 = d\phi^2 \tag{53b}$$

$$dl'^2 = F^{-2}(J)[dF^2(J) - J^2\, d\Omega^2(J)] \tag{53c}$$

respectively, then the composite mapping $d: M \to M'(J)$ is changed into a double mapping $d(J)$ as follows:



(54)

Therefore, the double composite mapping $d(J)$ can be expressed as

$$d(J) = k(J) \circ h \tag{55}$$

When $J = i$, $d(J) = d_C = k_C \cdot h$, which is the mapping from M to M'_C. When $J = \epsilon$, $d(J) = d_H = k_H \cdot h$, which is the mapping from M to M'_H. In addition, the requirement that $d(J)$ be harmonic means that h and $k(J)$ are both harmonic mappings. In terms of local coordinates the component of h is ϕ and those of $k(J)$ are $F(J)$ and $\Omega(J)$. The condition that h be a harmonic mapping is given as follows:

$$\phi_{\rho\rho} + \frac{1}{\rho} \phi_\rho + \phi_{zz} = 0 \tag{56}$$

whereas the condition that $k(J)$ be harmonic means $(F(J), \Omega(J))$ satisfies the geodesic equations on $M'(J)$, i.e.,

$$\begin{gathered} \frac{d^2F(J)}{d\phi^2} - \frac{1}{F(J)}\left[\left(\frac{dF(J)}{d\phi}\right)^2 + J^2\left(\frac{d\Omega(J)}{d\phi}\right)^2\right] = 0 \\ \frac{d^2\Omega(J)}{d\phi^2} - \frac{2}{F(J)} \frac{dF(J)}{d\phi} \cdot \frac{d\Omega(J)}{d\phi} = 0 \end{gathered} \tag{57}$$

which are equivalent to equations (40). Hence, the expressions for $F(J)$ and $\Omega(J)$ provide us with double solutions to the stationary axisymmetric gravitational equations. Evidently, the mapping d_C, which corresponds to the positive-definite metric of $M'(J)$, is just the composite mapping discussed by Eris (1977), but the mapping d_H, which corresponds to the negative-definite metric of $M'(J)$, is a new mapping.

5.2. The Double HJ Equation and Double Ernst Solution

Now we will solve the geodesic equations (57) by the HJ technique. For this purpose we first extend the HJ equation

$$\frac{\partial S}{\partial \phi} + H\left(Y^A, \frac{\partial S}{\partial Y^A}\right) = 0 \tag{58}$$

to a double form. Here ϕ and Y^A are local coordinates of M'' and $M'(J)$, respectively, S denotes the principal function, and the Hamiltonian H is simply the kinetic energy

$$H = \frac{1}{(g'')^{1/2}} g'^{AB} \frac{\partial S}{\partial Y^A} \frac{\partial S}{\partial Y^B} \tag{59}$$

where g'' and g' are the metrics of M'' and M', respectively. Combining equations (53b) and (53c) with equation (59), we have

$$H(J)=F^2(J)\left[\left(\frac{\partial S(J)}{\partial F(J)}\right)^2-J^2\left(\frac{\partial S(J)}{\partial \Omega(J)}\right)^2\right] \tag{60}$$

Thus, we obtain the double HJ equation as follows:

$$\frac{\partial S(J)}{\partial \phi}+F^2(J)\left[\left(\frac{\partial S(J)}{\partial F(J)}\right)^2-J^2\left(\frac{\partial S(J)}{\partial \Omega(J)}\right)^2\right]=0 \tag{61}$$

When $J=i$, equation (61) changes into

$$\frac{\partial S_C}{\partial \phi}+F_C^2\left[\left(\frac{\partial S_C}{\partial F_C}\right)^2+\left(\frac{\partial S_C}{\partial \Omega_C}\right)^2\right]=0 \tag{62a}$$

which is just the HJ equation given by Eris. When $J=\epsilon$, equation (61) changes into

$$\frac{\partial S_H}{\partial \phi}+F_H^2\left[\left(\frac{\partial S_H}{\partial F_H}\right)^2-\left(\frac{\partial S_H}{\partial \Omega_H}\right)^2\right]=0 \tag{62b}$$

This is called the hyperbolic HJ equation. Calculation gives the complete solution of equation (61),

$$\begin{aligned}S(J)=&-\beta(J)\phi+\alpha(J)\Omega(J)+[\beta(J)+J^2\alpha^2(J)F^2(J)]^{1/2}\\&-\beta^{1/2}(J)\ln\frac{\beta^{1/2}(J)+[\beta(J)+J^2\alpha^2(J)F^2(J)]^{1/2}}{\alpha(J)F(J)}\end{aligned} \tag{63}$$

where $\alpha(J)$ and $\beta(J)$ are both arbitrary double-real numbers. We easily find that S_C corresponding to $J=i$ is just the result given by Eris (1977). Further, in terms of the complex function

$$\xi(J)=\frac{1+F(J)+J\cdot\Omega(J)}{1-F(J)-J\cdot\Omega(J)} \tag{64}$$

the final result becomes

$$\xi(J)=-e^{J\theta}\operatorname{cth}\phi \tag{65}$$

where ϕ is a solution of the Laplace equation (56). It is clear that we once again obtain the double Ernst solution by using the double composite mapping and double HJ equation. This method not only extends Eris' results, but also indicates that there may be internal relations between the two different methods described in Sections 4 and 5, since both give the same result, i.e., the double Ernst solution. The investigation of this problem is the subject of forthcoming papers.

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